Math 10A with Professor Stankova
Worksheet, Discussion #11; Wednesday, 9/20/2017
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Graphing Functions

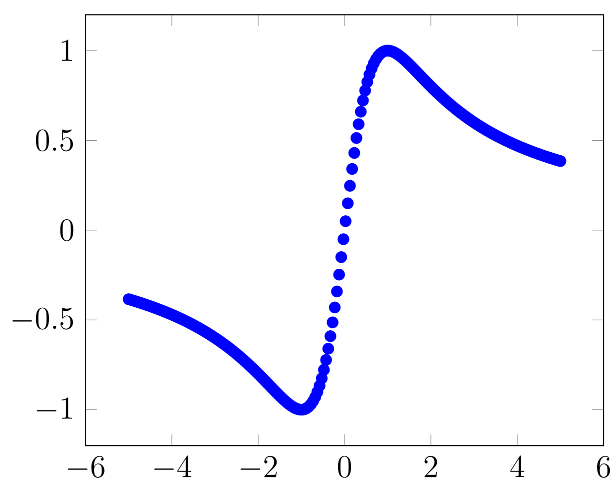
Example

1. Sketch the graph of $f(x) = \dfrac{2x}{x^2+1}$.

Solution: Take the derivative and second derivative to get $f'(x) = \dfrac{-2(x^2-1)}{(x^2+1)^2}$ and $f''(x) = \dfrac{4x(x^2-3)}{(x^2+1)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x^2 - 1 = 0$ so $x = \pm 1$, and $f''(x) = 0$ gives $x(x^2-3) = 0$. So the points we need to put in our table are $x = 0, \pm 1, \pm\sqrt{3}$. We fill out the table the sign of f', f'' on these intervals to get

	$< -\sqrt{3}$	$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	$\sqrt{3} <$
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$. So there is a horizontal asymptote at $y = 0$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = -1$ and maximum at $x = 1$ by the second derivative test.



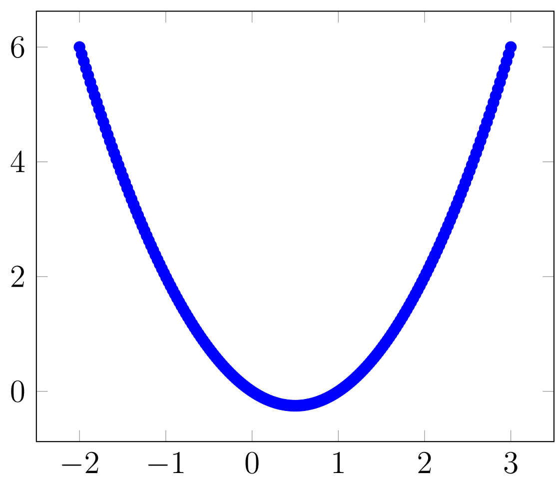
Problems

2. Sketch the graph of $f(x) = x^2 - x$.

Solution: Take the derivative and second derivative to get $f'(x) = 2x - 1$ and $f''(x) = 2$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x = 0.5$, and $f''(x) = 0$ has no solutions. So the point we need to put in our table is just $x = 0.5$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, 0.5)$	0.5	$(0.5, \infty)$
$f'(x)$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = 0.5$ by the second derivative test and f has a zero at $x = 0, 1$.

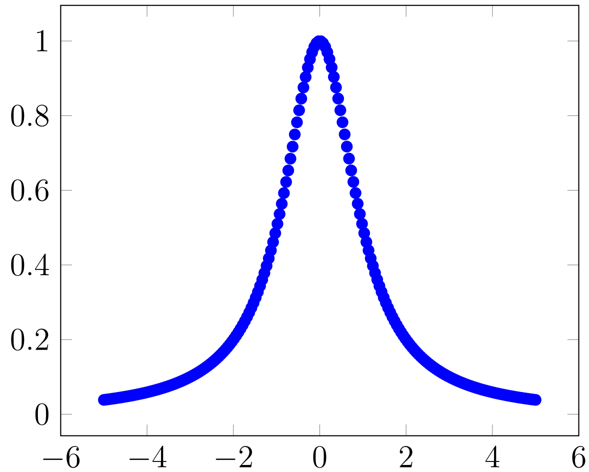


3. Sketch the graph of $f(x) = \dfrac{1}{1 + x^2}$.

Solution: Take the derivative and second derivative to get $f'(x) = \dfrac{-2x}{(x^2+1)^2}$ and $f''(x) = \dfrac{6x^2 - 2}{(x^2+1)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x = 0$, and $f''(x) = 0$ gives $6x^2 - 2 = 0$. So the points we need to put in our table are $x = 0, \pm 1/\sqrt{3}$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty,-1/\sqrt{3})$	$-1/\sqrt{3}$	$(-1/\sqrt{3},0)$	0	$(0,1/\sqrt{3})$	$1/\sqrt{3}$	$(1/\sqrt{3},\infty)$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$. So there is a horizontal asymptote at $y = 0$. We can now use this to produce something similar to the following graph noting that f will have a local maximum at $x = 0$ by the second derivative test.

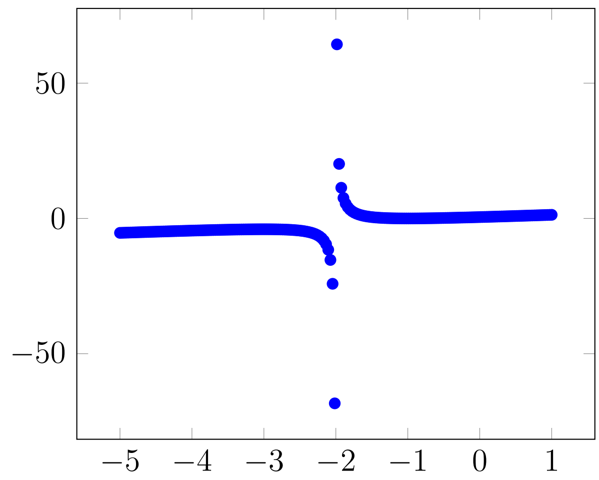


4. Sketch the graph of $f(x) = x + \dfrac{1}{2+x}$.

Solution: Take the derivative and second derivative to get $f'(x) = 1 - \dfrac{1}{(x+2)^2}$ and $f''(x) = \dfrac{2}{(x+2)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are not defined when $x = -2$ and solving $f'(x) = 0$ gives $(x+2)^2 = 1$ so $x = -3, -1$, and $f''(x) = 0$ has no solutions. So the points we need to put in our table are $x = -3, -2, -1$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty,-3)$	-3	$(-3,-2)$	-2	$(-2,-1)$	-1	$(-1,\infty)$
$f'(x)$	$+$	0	$-$	DNE	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	DNE	$+$	$+$	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$. Then since f is not defined at $x = -2$, we calculate the limits of f there with $\lim_{x\to-2^-} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$. So there is a vertical asymptote at $x = -2$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = -1$ and maximum at $x = -3$ by the second derivative test.

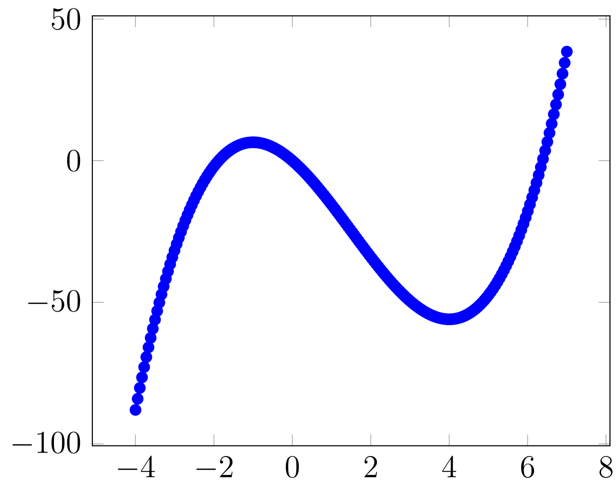


5. Sketch the graph of $f(x) = -12x - \frac{9x^2}{2} + x^3$.

Solution: Take the derivative and second derivative to get $f'(x) = -12 - 9x + 3x^2$ and $f''(x) = 6x - 9$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x^2 - 3x - 4 = 0$ so $x = -1, 4$, and $f''(x) = 0$ gives $x = 3/2$. So the points we need to put in our table are $x = -1, 3/2, 4$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, -1)$	-1	$(-1, 1.5)$	1.5	$(1.5, 4)$	4	$(4, \infty)$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = 4$ and maximum at $x = -11$ by the second derivative test.

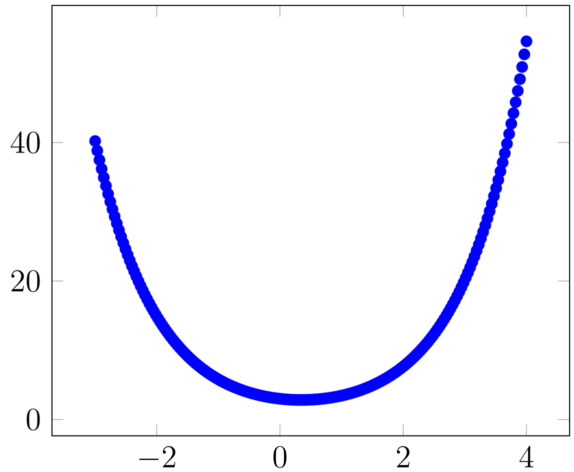


6. Sketch the graph of $f(x) = e^x + 2e^{-x}$.

Solution: Take the derivative and second derivative to get $f'(x) = e^x - 2e^{-x}$ and $f''(x) = e^x + 2e^{-x}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $e^{2x} = 2$ so $x = \frac{\ln 2}{2}$, and $f''(x) = 0$ has no solutions. So the point we need to put in is just $x = \frac{\ln 2}{2}$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, (\ln 2)/2)$	$(\ln 2)/2$	$((\ln 2)/2, \infty)$
$f'(x)$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = \frac{\ln 2}{2}$ by the second derivative test.

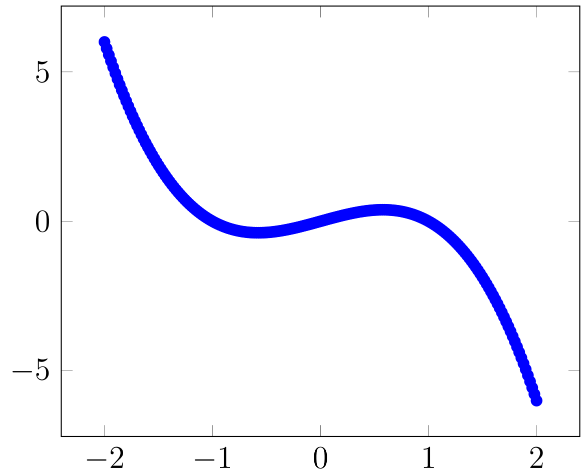


7. Sketch the graph of $f(x) = x - x^3$.

Solution: Take the derivative and second derivative to get $f'(x) = 1 - 3x^2$ and $f''(x) = -6x$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x^2 - 1/3 = 0$ so $x = \pm 1/\sqrt{3}$, and $f''(x) = 0$ gives $x = 0$. So the points we need to put in our table are $x = 0, \pm 1/\sqrt{3}$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, -1/\sqrt{3})$	$-1/\sqrt{3}$	$(-1/\sqrt{3}, 0)$	0	$(0, 1/\sqrt{3})$	$1/\sqrt{3}$	$(1/\sqrt{3}, \infty)$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = -1/\sqrt{3}$ and maximum at $x = 1/\sqrt{3}$ by the second derivative test.



8. Sketch the graph of $f(x) = \dfrac{x-3}{x+1}$.

Solution: Take the derivative and second derivative to get $f'(x) = \dfrac{4}{(x+1)^2}$ and $f''(x) = \dfrac{-2}{(x+1)^2}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are not defined at $x = -1$ and solving $f'(x) = 0, f''(x) = 0$ has no solutions. So the points we need to put in our table is just . We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, -1)$	-1	$(-1, \infty)$
$f'(x)$	$+$	DNE	$+$
$f''(x)$	$+$	DNE	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 1$. So there is a horizontal asymptote at $y = 1$. Now we calculate what happens as $x \to -1$ and we have $\lim_{x\to-1^-} f(x) = \infty$, $\lim_{x\to-1^+} f(x) = -\infty$ so it has a vertical asymptote at $x = -1$. We can now use this to produce something similar to the following graph.